Stochastic loop equations

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Abstract

Stochastic quantization is applied to derivation of the equations for the Wilson loops and generating functionals of the Wilson loops in the $N = \infty$ limit. These equations are treated both in the coordinate and momentum representations. In the first case the connection of the suggested approach with the problem of random closed contours and supersymmetric quantum mechanics is established, and the equation for the Quenched Master Field Wilson loop is derived. The regularized version of one of the obtained equations is presented and applied to derivation of the equation for the bilocal field correlator. The momentum loop dynamics is also investigated.

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1. Introduction

One of the possible ways of investigation of QCD is its reformulation in the space of all possible contours (the so-called loop space), when one considers gauge fields as chiral fields, defined on this space[1,2]. Loop equations (for a review see[3]) yield natural description of the confining phase of a gauge theory in terms of elementary excitations, which are nothing, but interacting closed strings[4], and the main purpose of these equations is to choose in the large-N limit those of the free string theories, which describe gauge fields. This problem is not completely solved up to now, though several attempts to this direction are done (see for example[5]). Loop equations are known to be nonlinear Schwinger equations in functional derivatives, formulated, generally speaking, for the generating functionals of the Wilson averages[6], and that is why the solution of these equations is a difficult mathematical problem, which nowadays is disentangled only in two dimensions[7] (notice also, that recently in 1+1 QCD the average of arbitrary number of Wilson loops on an arbitrary two-dimensional manifold was evaluated, and the relation to the string theory of maps was established[8]).

Recently alternative gauge-invariant equations were derived[9] using the stochastic quantization method[10] (for a review see[11]), and investigated in various field theories[12]. These equations are written not for Wilson loops, but for field correlators and therefore are closely connected to the Method of Vacuum Correlators[13,14] (for a review see[15]). The mathematical structure of the obtained equations is simpler than the structure of the loop equations, since the equations for correlators are just evolutionary integral-differential Volterra type-II ones. The derived system of equations for correlators is in some sense similar to the Bogolubov-Born-Green-Kirkwood-Ivon chain of equations for the Green functions in statistical mechanics[16], connecting correlators with various number of fields, and therefore the complete information about the field theory under consideration is contained only in the full infinite set of equations. In order to obtain a closed system of equations one needs to exploit some kind of approximation. The irreducible field correlators (the so-called cumulants [9,12,13,14,15,17] vanish, when any distance between two points, in which the fields in the cumulant are defined, increases and therefore it occurs natural to cut the obtained infinite chain of equations, using as a parameter of the approximation the number of fields in the cumulant (see discussion in[9]). For example, in[9,12] the so-called bilocal approximation[13,14,15], when all the cumulants higher than quadratic were neglected (which corresponded to the Gaussian stochastic ensemble of fields[17]), was exploited in order to investigate the obtained equations perturbatively both in the standard, non-gauge-invariant, and gauge-invariant ways, perform stochastic regularization, separate perturbative gluons′ contributions, include external matter fields and apply the suggested approach to quantization of classical solutions.

The aim of this paper is to apply the stochastic quantization method to derivation of equations for the Wilson loops in the $N = \infty$ limit alternative to the familiar loop equations. After that, using the cumulant expansion[9,12,13,14,15,17], one may get equations for correlators.

There are two types of equations for the Wilson loops, we are going to present. The equation of the first type, which will be investigated in Sections 2 and 3, is the evolutionary equation of the heat transfer form with the functional Laplacian standing for the ordinary one. This equation is true in the limit $N = \infty$ for the arbitrary values of the Langevin time. It will be derived in Section 2 for a single non-averaged Wilson loop and then generalized to the case of the generating functional for the Wilson loops. After that we smear the functional Laplacian in the obtained equation, using the method, suggested in[18], and reduce the problem to the integral Volterra type-II (by the Langevin time) equation. It will be shown that the action, over which

we shall perform the averaging in order to invert the smeared functional Laplacian (which is the action of the Euclidean harmonic oscillator at finite temperature[18]), after performing the limiting procedure, which restores the reparametrization invariance, has a Gaussian form, and therefore for the problem under consideration the Langevin equation, where the role of the Langevin time plays the parameter of a contour, may be written. Thus we come to the problem of random closed paths[2], which due to the fact that it may be described by the Langevin equation with the Langevin time being the proper time in the ensemble of contours is possible to be reduced[11] to the supersymmetric quantum mechanical problem[19], so that the dynamics of this system is governed by the two Fokker-Planck Hamiltonians[20,21], both of which survive (in contrast to the case considered in[20]), since one should not take the limit of the Langevin time tending to infinity. Also the equation for the Wilson loop of the Quenched Master Field[22] is derived in Section 2, and it is discussed that due to the discontinuities of the Wilson loop in the momentum representation[23] only the quenching prescription in the Langevin time direction yields nontrivial equation.

In Section 3 we rewrite the equation for the non-averaged Wilson loop in the momentum representation, using the methods suggested in[23,24]. We show that the role of the extra proper time introduced in[23], in which the Wilson loop propagates and splits, is played by the Langevin time.

In Section 4 we investigate the equation of the second type, which is also true only in the $N=\infty$ limit, but, in contrast to the equation of the first type, only in the asymptotical regime, when the Langevin time tends to infinity. This is the equation for the averaged Wilson loops, where the averaging is performed over the stochastic Gaussian noise fields, which is known to be equivalent at the Langevin time tending to infinity to the averaging with the gluodynamics action weight. However, it should be mentioned that since we work in the $N=\infty$ limit and neglect all the insertions of the stochastic Gaussian noise fields into the Wilson loop, this equation does not reproduce planar graphs. Up to now we do not know an equation for the averaged Wilson loops alternative to the familiar loop one, which reproduce correctly the planar graphs in the large-N limit, where the functional Laplacian may be replaced by the operator of differentiation by the Langevin time.

The equation obtained is then also generalized to the case of the generating functional[6] for the averaged Wilson loops, which provides Veneziano topological expansion[25], and rewritten in the momentum representation. Then we regularize the equation in the coordinate representation, using the method suggested in[26], after which it occurs to be defined on the space of smooth closed contours, to which one can apply the nonabelian Stokes theorem and the cumulant expansion. While integrated over the Langevin time t, this ultraviolet-regularized equation with the properly chosen t-dependent momentum cut-off yields the expression for the one-gluon-exchange diagram, where the perturbative gluon′s propagator is written in the Feynman-Schwinger path integral representation[14], and the Langevin time plays the role of the Schwinger proper time. Finally we use the obtained regularized equation in order to derive the equation for the bilocal correlator, whose Kronecker structure is supposed to be given.

The main results of the paper are summarized in the Conclusion.

2. Investigation of the heat transfer type equation in the coordinate representation

We shall start with the Langevin equation[10,11]

$$\dot{A}^a_\mu = (\nabla_\lambda F_{\lambda\mu})^a - ig\eta^a_\mu, \tag{1}$$

where $\nabla_\mu = \partial_\mu + [A_\mu, \cdot]$ is the adjoint covariant derivative, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is a strength tensor of the gluonic field, $g = \sqrt{\frac{\lambda}{N}}$, where λ is the bare coupling, which remains finite while performing the $\frac{1}{N}$ expansion, and η^a_μ is a stochastic Gaussian noise field.

Let us introduce the following loop functional of the Stokes type

$$\Psi_\eta(C,t) = \frac{1}{N} trT\ Pexp \oint\limits_C dz_\mu \left(A_\mu(z,t) + ig \int\limits_0^t dt' \eta_\mu(z,t') \right) \equiv$$

$$\equiv \frac{1}{N} tr \lim_{\Delta t_i, \Delta z_j \to 0} \prod_i \left(1 + \Delta t_i \sum_j (\Delta z_j)_\mu (2A_\mu(z_j,t_i)\delta(t_i - t) + ig\eta_\mu(z_j,t_i)) \right).$$

Here in addition to the usual P-ordering we have introduced T-ordering because of the second term in the exponent, which contains t-integration.

Differentiating $\Psi_\eta(C,t)$ by the Langevin time one gets by virtue of (1) the following equation

$$\dot{\Psi}_\eta(C,t) = \frac{1}{N} tr \oint\limits_C dx_\mu (\nabla_\lambda F_{\lambda\mu}(x)) T\ Pexp \oint\limits_C dz_\mu \left(A_\mu(z,t) + ig \int\limits_0^t dt' \eta_\mu(z,t') \right) =$$

$$= \frac{1}{N} tr \oint\limits_C dx_\mu (\nabla_\lambda F_{\lambda\mu}(x)) \Bigg(\phi(C,t) + ig \oint\limits_C dz_\mu \phi(C_{x_0 z},t) \int\limits_0^t dt' \eta_\mu(z,t') \phi(C_{z x_0},t) -$$

$$- g^2 \oint\limits_C dz_\mu \oint\limits_C dy_\nu \phi(C_{x_0 z},t) \int\limits_0^t dt' \eta_\mu(z,t') \phi(C_{zy},t) \int\limits_0^t dt'' \eta_\nu(y,t'') \phi(C_{y x_0},t) + \ldots \Bigg), \tag{2}$$

where $\phi(C_{xy},t) \equiv P\ exp \int\limits_{C_{xy}} dz_\mu A_\mu(z,t)$, and x_0 is an arbitrary but fixed point belonging to the contour C.

It was proved in[6] that the $\frac{1}{N}$ expansion corresponded to the WKB approximation around the "classical" field in the effective scalar theory in the loop space. We see that this is really the case according to equation (2) since in the stochastic quantization method[10,11] the degree of quantum correction of the grand ensemble of fields A^a_μ and η^a_μ is determined through the maximal number of the noise fields entering the physical quantity under consideration, and on the right-hand side of equation (2) any new term of the $\frac{1}{N}$ expansion gives rise to the additional power of the noise field.

In what follows we shall derive and investigate the equation for the pure "classical" field in the loop space $\Psi(C,t) = \frac{1}{N} tr\phi(C,t)$ corresponding to $N = \infty$. This equation reads

$$\dot{\Psi}(C,t) = \oint\limits_C dx_\mu \partial^x_\lambda \frac{\delta}{\delta\sigma_{\lambda\mu}(x)} \Psi(C,t), \quad t < +\infty, \tag{3}$$

where $\partial^{x(\sigma)}_\lambda \equiv \int\limits_{\sigma-0}^{\sigma+0} d\sigma' \frac{\delta}{\delta x_\lambda(\sigma')}$.

Notice that equation (3) is true for an arbitrary $t < +\infty$, and from the mathematical point of view it is a homogeneous functional heat transfer equation (which means that the loop space

Laplacian stands there for the ordinary one), while the loop equation in the $N = \infty$ limit is an inhomogeneous functional Laplace one.

Using equation (3) one can easily write an equation for the generating functional of the Wilson loops

$$Z(J(C)) = exp\left(\sum_C J(C)\Psi(C)\right),$$

where the sum over the loops is defined as follows[6]

$$\sum_C f(C) = \int_{+0}^{+\infty} \frac{dT}{T} \int_{x(0)=x(T)} Dx(\alpha) f(x(\alpha)).$$

This equation reads

$$\frac{\partial}{\partial t}\frac{\delta Z}{\delta J(C)} = \oint_C dx_\mu \partial^x_\lambda \frac{\delta}{\delta\sigma_{\lambda\mu}(x)}\frac{\delta Z}{\delta J(C)} + \sum_{C'} J(C') \oint_{C'} dz_\mu \partial^z_\lambda \frac{\delta}{\delta\sigma_{\lambda\mu}(z)}\frac{\delta^2 Z}{\delta J(C)\delta J(C')} \tag{4}$$

and does not depend on N explicitly. We shall see in Section 4 that this dependence appears in the equation for the generating functional of the averaged Wilson loops, which describes an "open string" with quarks at the ends. This equation will correspond to the Veneziano topological expansion[6,25].

Let us now perform the smearing of the functional Laplacian, standing on the right-hand side of equation (3), by making use of the method suggested in[18]. It may be done via performing the polygon discretization procedure and then going to the continuum limit or directly using the continuum smearing without any reference to the polygon discretization. This smearing implies that one replaces the functional Laplacian

$$\Delta \equiv \oint_C dx_\mu \partial^x_\lambda \frac{\delta}{\delta\sigma_{\lambda\mu}(x)} = \int_0^1 d\sigma \int_{\sigma-0}^{\sigma+0} d\sigma' \frac{\delta}{\delta x_\mu(\sigma')}\frac{\delta}{\delta x_\mu(\sigma)}$$

with the smeared one

$$\Delta^{(G)} = \int_0^1 d\sigma \quad v.p. \int_0^1 d\sigma' G(\sigma - \sigma') \frac{\delta}{\delta x_\mu(\sigma')}\frac{\delta}{\delta x_\mu(\sigma)} + \Delta, \tag{5}$$

where $G(\sigma - \sigma')$ is a smearing function, and the first term on the right-hand side of equation (5), which involves the principal-value integral, is an operator of the second order (does not satisfy the Leibnitz rule in contrast to the operator Δ) and reparametrization noninvariant. The averaging over the loops $\xi(\sigma)$ is defined as follows

$$< F[\xi] >^{(G)}_\xi = \frac{\int\limits_{\xi(0)=\xi(1)} D\xi e^{-S} F[\xi]}{\int\limits_{\xi(0)=\xi(1)} D\xi e^{-S}}, \tag{6}$$

where

$$S = \frac{1}{2}\int\limits_0^1 d\sigma \int\limits_0^1 d\sigma' \left(\xi(\sigma)G^{-1}(\sigma-\sigma')\xi(\sigma')\right), \tag{7}$$

and G^{-1} is an inverse operator. For the simplest case

$$G(\sigma-\sigma') = e^{-\frac{|\sigma-\sigma'|}{\varepsilon}}, \quad \varepsilon \ll 1$$

the action (7) reduces to the action of the Euclidean harmonic oscillator at finite temperature[18]

$$S = \frac{1}{4}\int\limits_0^1 d\sigma \left(\varepsilon\dot{\xi}^2(\sigma) + \frac{1}{\varepsilon}\xi^2(\sigma)\right), \tag{8}$$

which may be recovered from the discretized action, when the number of vertices of the polygon, which approximates the loop, tends to infinity. The contribution of the first term on the right-hand side of equation (5) is of the order ε for smooth contours since the region $|\sigma-\sigma'| \sim \varepsilon$ is essential in the integral over $d\sigma'$. Hence when ε tends to zero, this term vanishes, reparametrization invariance restores, and $\Delta^{(G)} \to \Delta$.

Thus equation (3) takes the form

$$\Delta^{(G)}\Psi[x,t] = \dot{\Psi}[x,t], \tag{9}$$

where $x = x(\sigma)$ is the position vector of the contour C. Using the equation of motion

$$< \xi_\mu(\sigma)F[\xi] >_\xi^{(G)} = \int\limits_0^1 d\sigma' G(\sigma-\sigma') < \frac{\delta F[\xi]}{\delta\xi^\mu(\sigma')} >_\xi^{(G)},$$

which follows from the oscillator action (8), one can solve equation (9)[18] with the condition $\Psi[0,t] = 1$. Integrating over the Langevin time we arrive at the following equation

$$\int\limits_0^t dt'\Psi[x,t'] + \frac{1}{2}\int\limits_0^{+\infty} d\gamma \left(< \Psi[x+\sqrt{\gamma}\xi,t] >_\xi^{(G)} - < \Psi[\sqrt{\gamma}\xi,t] >_\xi^{(G)}\right) =$$

$$= t + \frac{1}{2}\int\limits_0^{+\infty} d\gamma \left(< \Psi[x+\sqrt{\gamma}\xi,0] >_\xi^{(G)} - < \Psi[\sqrt{\gamma}\xi,0] >_\xi^{(G)}\right), \tag{10}$$

which is the integral Volterra type-II one by the Langevin time, and the dependence on the initial conditions is put to the right-hand side.

Finally let us go to the limit $\varepsilon \to 0$ in order to restore the reparametrization invariance of equation (10). Then performing the rescaling $\zeta = \frac{\xi}{\sqrt{\varepsilon}}$, we see that the averaging in both sides of equation (10) becomes Gaussian with the action $S = \frac{1}{4}\int\limits_0^1 d\sigma\zeta^2(\sigma)$. It means that we come to the problem of motion of random closed paths, which occurs to be described by the Langevin equation, where the role of the Langevin time plays the parameter of the contour (i.e. the proper time in the loop space). The problem of random closed contours was investigated in[2], where it was shown that the number of contours of the length T was given by the formula

$$dN(T) = \frac{dT}{T}T^{-\frac{D}{2}}e^{-\frac{cT}{\epsilon}},$$

where D was the dimension of the space-time, c was some constant, and ϵ was a cut-off parameter.

Within our approach we reduced this problem to the Langevin equation, written in the proper time of the system under consideration. Therefore it is equivalent[11] to the supersymmetric quantum mechanical problem[19]. Namely if one writes down in components the equation of the Brownian motion of a classical particle in a heat bath, which describes the evolution of the position vector of a contour,

$$\frac{\partial x}{\partial \sigma} = -\frac{\delta S}{\delta x} + \eta,$$

then making the change of variables $\eta \to x$ in the partition function for the Langevin dynamics, we obtain the following effective Lagrangian

$$\mathcal{L}_{eff.} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{x}V + \frac{1}{8}V^2 - \bar{\psi}\left(\frac{\partial}{\partial\sigma} + \frac{1}{2}V'\right)\psi,$$

where " $\cdot$ " $\equiv \frac{\partial}{\partial\sigma}$, " $\prime$ " $\equiv \frac{\delta}{\delta x}$, $V \equiv S'$. In order to have a closed supersymmetry algebra, one needs to introduce an auxiliary field D:

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 - \frac{1}{2}D^2 - \frac{1}{2}DV - \bar{\psi}\left(\frac{\partial}{\partial\sigma} + \frac{1}{2}V'\right)\psi, \tag{11}$$

so that the action, corresponding to (11), is invariant under the supersymmetry transformations

$$\delta x = \bar{\varepsilon}\psi - \bar{\psi}\varepsilon, \quad \delta D = \bar{\varepsilon}\dot{\psi} + \dot{\bar{\psi}}\varepsilon, \quad \delta\psi = (\dot{x} + D)\varepsilon, \quad \delta\bar{\psi} = \bar{\varepsilon}(\dot{x} - D).$$

The dynamics of the system may be shown[20,21] to be governed by the two Fokker-Planck Hamiltonians

$$H^{\pm} = -\frac{1}{2}\frac{\delta^2}{\delta x^2} + \frac{1}{8}V^2 \pm \frac{1}{4}V',$$

where the Hamiltonian H^- corresponds to the propagation forward in the Langevin time σ, and H^+ corresponds to the backward propagation. The existence of the two Fokker-Planck Hamiltonians is a consequence of the supersymmetry. However it should be emphasized that while in the usual causal interpretation of the Langevin equation, which requires propagation forward in the Langevin time, in the equilibrium limit only the forward dynamics survives since only the zero ground state of H^- contributes, and H^+ has strictly positive eigenvalues[20], in our model it is not the case, because one should not take the limit $\sigma \to +\infty$. Therefore the dynamics of the system is governed by both the forward and backward Hamiltonians.

To conclude this Section, we obtain the equation for the Quenched Master Field[22] Wilson loops. The important property of the Wilson loops in the momentum representation is the presence of finite discontinuities induced by the emission and absorption of gluons[23]

$$\Delta p_\mu(s_i) = p_\mu(s_i + 0) - p_\mu(s_i - 0),$$

and therefore only quenching in the Langevin time direction is valid

$$A_\mu^{ab}(x,t) = e^{i(p_{5a}-p_{5b})t}\bar{A}_\mu^{ab}(x), \quad \eta_\mu^{ab}(x,t) = e^{i(p_{5a}-p_{5b})t}\bar{\eta}_\mu^{ab}(x),$$

where

$$< \bar{\eta}^{ab}_{\mu}(x)\bar{\eta}^{cd}_{\nu}(y) >= 2\frac{\Lambda_5}{2\pi}\delta^{bc}\delta^{ad}\delta_{\mu\nu}\delta(x-y).$$

Introducing a new loop functional

$$\Omega(C,t) = \frac{1}{N} tr \oint_C dx_\mu P \left(B_\mu(x,t) exp \oint_C dy_\lambda A_\lambda(y,t) \right),$$

where $B^{ab}_{\mu}(x,t) = i(p_{5a} - p_{5b})A^{ab}_{\mu}(x,t)$, we obtain from (3), performing the smearing of the functional Laplacian as it has been done above, the following equation for the loop functionals Ψ and Ω:

$$\Psi[x,t] = 1 - \frac{1}{2}\int_0^{+\infty} d\gamma \left(< \Omega[x + \sqrt{\gamma}\xi, t] >^{(G)}_{\xi} - < \Omega[\sqrt{\gamma}\xi, t] >^{(G)}_{\xi} \right).$$

3. Heat transfer type equation in the momentum representation

In this Section we shall rewrite equation (3) in the momentum representation in order to investigate free momentum loop dynamics in the Langevin time. During this free motion the loop behaves as a collection of free particles, while the loops′ collisions, at which momenta are rearranged between particles, which means that the loops split, will be considered in the next Section.

The loop functional in the momentum representation is defined as follows[23,24]:

$$\Psi(P,t) = \int DC exp\left(i \int_C p_\mu dx_\mu \right) \Psi(C,t).$$

Using the representation

$$\Delta = \int_0^1 ds_1 \int_{s_1-0}^{s_1+0} ds_2 \frac{\delta^2}{\delta x_\mu(s_1)\delta x_\mu(s_2)}$$

for the functional Laplacian and following the procedures suggested in[23,24], one gets from equation (3) two equations, which describe the free propagation of the momentum loop:

$$\dot{\Psi}(P,t) = -\sum_i (\Delta p_\mu(s_i))^2 \Psi(P,t), \tag{12}$$

$$\dot{\Psi}(P,t) = -2\int_0^1 ds_1 p'_\mu(s_1) \int_0^{s_1} ds_2 p'_\mu(s_2)\Psi(P,t).$$

Notice that in equation (12) the reparametrization invariance is preserved, i.e. when the parameter s changes to $f(s)$, where $f'(s) > 0$, the positions of the discontinuities s_i shift, but their order and the values of $\Delta p_\mu(s_i)$ remain the same.

Therefore one may conclude that the role of the extra proper time H, which was introduced in[23], in which the Wilson loop propagates and bits, plays the Langevin time t. This evolution is however alternative to the evolution of the loop in the space of random contours, described by the

supersymmetric quantum mechanics (see the previous Section), where the role of the proper time played the parameter of the contour.

4. An equation for the averaged Wilson loops

This Section is devoted to investigation of the equation alternative to equation (3), which will be written for the averaged Wilson loops. It can be derived by averaging both sides of equation (3) over the stochastic noise fields η^a_μ, which in the physical limit of the Langevin time tending to infinity is equivalent[10,11] to the averaging with the gluodynamics action weight. Therefore in the asymptotical regime $t \to +\infty$ one may use the equation of motion for the Wilson loop $\Psi(C,t)$, which is nothing, but the loop equation in the limit $N = \infty$. Thus we obtain the following equation:

$$\dot{\Phi}(C,t) = \lambda \oint_C dx_\mu \;\; v.p. \int_C dy_\mu \delta(x-y)\Phi(C_{xy},t)\Phi(C_{yx},t), \quad t \to +\infty, \tag{13}$$

where $\Phi(C,t) =< \Psi(C,t \to +\infty) >_{\eta^a_\mu}$, and the principal value integral on the right-hand side of equation (13) implies that we integrate only over those y_μ 's, which are another points of the contour C as the point x. In other words the right-hand side of equation (13) does not vanish only when the contour C has self-intersections.

As it was already mentioned in the Introduction, this equation does not reproduce the planar graphs since we have neglected insertions of the stochastic noise fields into the Wilson loop in the limit $N = \infty$. It should be understood as an asymptotical at t tending to infinity equation for the "classical" Wilson average.

Equation (13) may be generalized to the case of the generating functional $\mathcal{Z}(C,j) = \frac{1}{N}\frac{\delta \; ln<Z>}{\delta J(C)}$, which is also a functional of the source $j(C) = \frac{J(C)}{N}$. If N tends to infinity, but the ratio $\rho = \frac{N_f}{N}$ is fixed, which corresponds to the Veneziano topological expansion[25], generalizing the $\frac{1}{N}$ expansion, then the source j remains finite. The equation for the functional $\mathcal{Z}$ reads

$$\frac{\partial}{\partial t}\mathcal{Z}(C) = \lambda \oint_C dx_\mu \;\; v.p. \int_C dy_\mu \delta(x-y)\left(N\mathcal{Z}(C_{xy})\mathcal{Z}(C_{yx}) + \frac{1}{N}\frac{\delta \mathcal{Z}(C_{xy})}{\delta j(C_{yx})}\right) +$$
$$+ \sum_{C'} j(C') \oint_{C'} dz_\mu \partial^z_\lambda \frac{\delta}{\delta \sigma_{\lambda\mu}(z)}\left(N^2 \mathcal{Z}(C)\mathcal{Z}(C') + \frac{\delta \mathcal{Z}(C)}{\delta j(C')}\right).$$

This is the Schwinger equation in functional derivatives, which describes the evolution of the wave functional of the open string with quarks at the ends (see discussion in[6]).

Let us now rewrite equation (13) in the momentum representation. It may be done using the procedure, suggested in[23], and the answer has the form

$$\dot{\Phi}(P,t) = -2\lambda \int_0^1 ds_2 \frac{\delta}{\delta p_\mu(s_2)} \int_0^{s_2} ds_1 \frac{\delta}{\delta p_\mu(s_1)} \Phi(P^{(1)},t)\Phi(P^{(2)},t), \tag{14}$$

where $P^{(1)}$ and $P^{(2)}$ are the parts of P from s_1 to s_2 and from s_2 to s_1 respectively:

$$P^{(1)}: \;\; p^{(1)}_\mu(s) = p_\mu((1-s_{21})s), \; 0 < s < s', \;\; p^{(1)}_\mu(s) = p_\mu((1-s_{21})s + s_{21}), \; s' < s < 1, \tag{15}$$

$$P^{(2)}: \ p_\mu^{(2)}(s) = p_\mu(s_1 + s_{21}s), \ 0 < s < 1, \tag{16}$$

where $s_{21} = s_2 - s_1, \ s' = \frac{s_1}{1-s_{21}}$.

Equation (14) describes the vertex of the splitting of the loop into two loops $P \to P^{(1)} + P^{(2)}$, so that the momentum $p_\mu(s)$ of the initial loop is distributed between these two new loops according to the formulae (15) and (16). As was discussed in[23], at every splitting the effective number of degrees of freedom per loop diminishes, so that if one approximates the initial loop by a polygon, the number of vertices of each of the polygons, approximating the loops after the splitting, will be smaller. At each splitting of the loop the function $p_\mu(s)$ smothers, and the derivative $p'_\mu(s)$ decreases. Finally the loop, propagating in the Langevin time and splitting during this process, runs out of degrees of freedom and reduces to the point in the momentum space $p_\mu(s) = p_\mu = const$.

Our next aim is to apply to equation (13) the nonabelian Stokes theorem and the cumulant expansion. After that using the bilocal approximation[13,14,15], we shall derive the equation for the bilocal correlator, whose Kronecker structure will be supposed to be given. To this end we perform an ultraviolet regularization of equation (13), exploiting the method, suggested in[26], where this procedure was done with the help of the heat-kernel regularized Langevin equation. The regularized version of equation (13) has the form

$$\dot{\Phi}(C,t) = \lambda \oint_C dx_\mu \oint_C dy_\mu \int_x^y Dr exp \left(-\frac{1}{4} \int_0^{\Lambda^{-2}} ds \dot{r}^2(s) \right) \Phi(C_{xy} r_{yx}, t) \Phi(C_{yx} r_{xy}, t), \tag{17}$$

where $r(0) = x, \ r(\Lambda^{-2}) = y$, and the momentum cut-off Λ may be taken for example to be of the order of the inverse correlation length of the vacuum[15] $T_g \simeq 0.2 fm : |\Lambda| \sim \frac{1}{T_g}$. However in what follows we shall use as a momentum cut-off $\frac{1}{\sqrt{t}}$.

It is known[27] that in the stochastic quantization of gauge theories it is not necessary to add a gauge-fixing term into the Langevin equation, since direct iterations in powers of the coupling constant without introducing ghost fields yield the same results as the Faddeev-Popov perturbation theory, because the Langevin time t takes the role of a gauge parameter. If one fixes t, calculates gauge-invariant quantities and then goes to the physical limit $t \to +\infty$, all the lineary divergent in t terms will cancel each other in the same way as the terms, depending on the gauge parameter in the framework of the usual approach.

Therefore our choise of the momentum cut-off looks natural from the point of view of the general principa of the stochastic quantization method. If we now integrate equation (17) by the Langevin time, analitically continuing both sides down from the asymptotics $t \to +\infty$, then the obtained equation yields the expression for the second order one-gluon-exchange diagram in the perturbation theory in the nonperturbative gluodynamics vacuum[14], where gluonic propagator is written in the Feynman-Schwinger path integral representation, and the role of the Schwinger proper time plays the Langevin time. This perturbative gluon propagates along a regulator path r_{xy}, dividing the surface, swept by the nonperturbative gluodynamics string, into two pieces.

Equation (17) is defined on the space of closed smooth contours, and therefore one can apply to it the nonabelian Stokes theorem and the cumulant expansion, after which the integral over the regulator paths, standing on the right-hand side of this equation, yields the amplitude of propagation of a free gluon. Let us vary both sides of equation (17), to which we applied the cumulant expansion in the bilocal approximation, by the element of the surface, lieing on the

contour C, and suppose for simplicity that the bilocal correlator has the only one Kronecker structure:

$$< F_{\mu\nu}(x,t)\phi(\Gamma_{xy},t)F_{\lambda\rho}(y,t)\phi(\Gamma_{yx},t) >= \frac{\hat{1}}{N}(\delta_{\mu\lambda}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\lambda})D((x-y)^2,t),$$

where Γ_{xy} and Γ_{yx} are some contours. Then we get the following equation for the correlation function D:

$$\int\limits_S d\sigma_{\lambda\rho}(x_2)D((x_1-x_2)^2,t) = \frac{g^2}{8\pi^2}\frac{\delta}{\delta\sigma_{\lambda\rho}(x_1)}\oint\limits_C dx_\mu \oint\limits_C dy_\mu \frac{e^{-\frac{(x-y)^2}{4t}}}{(x-y)^2},$$

where the point x_1 belongs to the contour C, the surface S is bounded by this contour, and the initial condition $D(x^2,0)=0$ is implied.

5. Conclusion

In this paper we applied the stochastic quantization method to derivation of the equations for Wilson loops in the $N=\infty$ limit. There are two types of equations: first is the heat transfer type equation (3), which is true for arbitrary values of the Langevin time. This is the evolutionary equation for the non-averaged Wilson loop. It was derived in Section 2 and then generalized to the case of the generating functional for the Wilson loops. After that, applying the smearing procedure to the functional Laplacian which stands on the right-hand side of this equation, we reduced the problem to the Volterra type-II integral equation (by the Langevin time) and established its connection with the problem of random closed paths. The latter occured to be described by the supersymmetric quantum mechanics of a particle in a heat bath, where the role of the proper time played the parameter of a contour. Therefore the dynamics of the system of random contours is governed by the forward and backward Fokker-Planck Hamiltonians.

The quenching prescription may be applied to the obtained equation only in the Langevin time direction, since in the momentum representation the Wilson loop possesses discontinuities. The equation for the Wilson loop of the Quenched Master Field is presented at the end of Section 2.

In Section 3 the heat transfer type equation was investigated in the momentum representation, and two alternative equations, describing the free propagation of the non-averaged Wilson loop in the Langevin time, were derived.

In Section 4 an alternative equation for the averaged Wilson loops, which was also true in the limit $N=\infty$ but in the asymptotical regime of the large values of the Langevin time t, was derived and generalized to the case of the generating functional for the Wilson averages. This equation for the averaged Wilson loops, while rewritten in the momentum representation, describes splitting of the loop during its propagation in the Langevin time. At these splittings the loop loses its degrees of freedom and finally shrinks to a point in the momentum space.

Then in order to apply to the equation in the coordinate representation the nonabelian Stokes theorem and the cumulant expansion and derive the equation for the bilocal cumulant, we regularized this equation, using the method suggested in[26], after which, choosing for the momentum cut-off $\frac{1}{\sqrt{t}}$ and integrating over t, one may recognize in the obtained expression the one-gluon-exchange diagram[14], contributing to the averaged Wilson loop. Applying to the both sides of this equation the nonabelian Stokes theorem and the cumulant expansion and assuming that the

bilocal correlator had a given Kronecker structure, we obtained the equation for the corresponding coefficient function.

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